Estimation of Q-factor and phase velocity using the
recovered stress-strain relaxation spectrum

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Summary
The paper presents a numerical inversion method for estimation of Q-factor and phase velocity
in viscoelastic media using recovery of relaxation spectrum from measured or computed
complex velocity or complex modulus of the medium. Mathematically the problem is formulated
as an inverse problem for reconstruction of spectral measure in the Stieltjes representation of
the complex modulus using rational ([p,q]-Padé) approximation. The approximation is obtained
by solving a constrained least squares minimization problem with regularization. The recovered
stress-strain relaxation spectrum is applied to numerical calculation of frequency dependent Q-
factor and frequency dependent phase velocity for a standard linear viscoelastic solid (Zener)
model as well as a nearly constant-Q model which has a continuous spectrum. Numerical
results show good agreement between theoretical and predicted values and demonstrate the
validity of the algorithm. The method can be used for evaluating relaxation mechanisms in
seismic wavefield simulation of viscoelastic media. The constructed lower order Padé
approximation can be used for determination of the internal memory variables in time-domain
finite difference (TDFD) numerical simulation of viscoelastic wave propagation.

Introduction
The inversion method of Padé approximation is based on a constrained least squares
minimization algorithm, regularized by the constraints derived from the analytic Stieltjes integral
representation of the complex modulus. Solution of the constrained minimization problem
provides coefficients of a rational approximation to the spectral measure of the medium. This
rational approximation is transformed into Padé approximation by partial fraction decomposition.
The method can use as data the values of measured, or simulated complex modulus or
complex velocity in certain interval of frequencies. The recovered lower order [p,q]-Padé
approximation can be used for determination of the internal memory variables in TDFD
numerical simulation of viscoelastic wave propagation. The developed technique together with
finite difference modeling may eventually lead to a new simultaneous inversion technique for
estimation of the frequency dependent complex velocities, Q-factors and phase velocities in
anelastic attenuating media from vertical seismic profile (VSP) data in geophysics prospecting.
**Q-factor modeling and rational approximation for inversion**

We consider a plane compressional wave propagating in a homogeneous isotropic viscoelastic medium with constant material properties. The equation of motion and the relation between stress $\sigma$ and strain $\varepsilon$ for one-dimensional (1D) linear viscoelastic media are represented by

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \quad \sigma = M * d\varepsilon = \int_{-\infty}^{t} M(t-\tau)d\varepsilon(\tau), \quad \varepsilon = \frac{\partial u}{\partial x}$$

where $\rho$ is the mass density, $u(x,t)$ is the displacement, $M(t)$ is the relaxation function of the medium. In the frequency domain, the relation between stress $\sigma$ and strain $\varepsilon$ in a linear viscoelastic medium and the Q-factor can be formulated as

$$\sigma(\omega) = M(\omega)\varepsilon(\omega), \quad Q(\omega) = \text{Re} M(\omega)/\text{Im} M(\omega) = 1/\tan \theta(\omega)$$

where $M(\omega)$ is the complex viscoelastic modulus, $\theta$ is the phase of $M$. The complex velocity $V(\omega)$ and the phase velocity $c(\omega)$ in an attenuating medium are given by (Carcione, 2007):

$$V(\omega) = \sqrt{M(\omega)/\rho}, \quad 1/c(\omega) = \text{Re} \sqrt{\rho/M(\omega)}$$

respectively. $M(\omega)$ is uniquely determined by a given $Q(\omega)$ in a causal medium since $\text{Re} M$ and $\text{Im} M$ must obey a Kramers-Kronig relation. In seismic applications, $Q(\omega)$ is normally assumed to be frequency-independent or only slowly varying with frequency. The information about the relaxation spectrum of the medium is contained in the spectral measure $\eta(x)$ in the analytic Stieltjes integral representation of the complex modulus (Day & Minster, 1984):

$$G(s) = \frac{M_U - M(s/i)}{\delta M} = \int_0^{\infty} \frac{d\eta(x)}{s + x}, \quad \text{with the sum rule} \int_0^{\infty} \frac{d\eta(x)}{x} = 1, \ s = i\omega.$$  

Here $i = \sqrt{-1}$, $M_U$ is the unrelaxed modulus and $\delta M$ is the relaxation of the modulus. The function $G(s)$ is analytic outside the negative real semi axis in the complex s-plane, all its singular points are in the interval $(-\infty,0)$. The function $\eta(x)$ can be approximated by a step function with a finite number of steps, so that

$$d\eta(x) \equiv d\hat{\eta}(x) = \sum_{n=1}^{q} A_n \delta(x+s_n)dx, \quad x \in (0,\infty).$$

Thus, the approximation $\hat{G}(s)$ of the function $G(s)$ is given by

$$G(s) \approx \hat{G}(s) = \sum_{n=1}^{q} \frac{A_n}{s - s_n}, \quad \text{s. t.} \quad -\infty < s_n < 0, \quad 0 < \frac{A_n}{|s_n|} < 1, \quad \sum A_n = 1.$$  

Here $s_n$ is the n-th simple pole on the negative real semi axis with positive residue $A_n$, $q$ is the total number of poles. The approximation of the complex modulus $M(\omega)$ is obtained as

$$M(\omega) \approx M_U - \delta M \sum_{n=1}^{q} \frac{A_n}{i\omega - s_n}.$$  

Therefore, the Q-factor and $V(\omega)$ can be estimated in terms of $A_n$ and $s_n$ as

$$Q(\omega) \approx \frac{\text{Re} \{M_U - \delta M \sum A_n/(i\omega - s_n)\}}{\text{Im} \{M_U - \delta M \sum A_n/(i\omega - s_n)\}}, \quad V(\omega) \equiv V^c(\omega) = \frac{1}{\sqrt{\rho}} \left( M_U - \delta M \sum_{n=1}^{q} \frac{A_n}{i\omega - s_n} \right)^{1/2}.$$  

Then the phase velocity can be estimated as $c(\omega) \equiv 1/\text{Re} V^c(\omega)$. A new numerical inversion
algorithm for reconstruction of the measure \( \eta(x) \) is developed using constrained \([p,q]-Padé\) approximation (Baker, 1996) of \( \hat{G}(s) \) and its partial fraction decomposition. The approximation has the form of \( G(s) \equiv \hat{G}(s) = a_p(s) / b_q(s) \), \( p < q \); \( p \), \( q \) are the orders of real polynomials \( a_p(s) \) and \( b_q(s) \), respectively. We assume that the complex velocity \( V(\omega) \) or complex modulus \( M(\omega) \) can be measured at sample data point of frequencies \( \omega_k \) \( (k = 1,2,\cdots,N) \) where \( N \) is the total number of data points. Measurements \( V(\omega_k) \) or \( M(\omega_k) \) can be transformed to \( G(z_k) \) in the complex \( s \)-plane, \( z_k = i\omega_k \), corresponding to each sample frequency. Thus we have data pairs \( (z_k, d_k) \), \( d_k = G(z_k) \). The unknown coefficients of \( a_p(s) \) and \( b_q(s) \) are determined by solving the linear system of equations: \( Sc = g \). Here the vector \( c \) contains all normalized coefficients of \( a_p(s) \) and \( b_q(s) \), \( g = g_R + ig_I \) and \( S = S_R + iS_I \), subindices \( R \) and \( I \) indicating the real and imaginary parts of the matrices with entries in terms of data. The reconstruction problem of determining the coefficient vector \( c \) is an inverse problem; it is ill-posed. To derive a stable numerical inversion algorithm, a penalization term was introduced in the Tikhonov regularization functional (Tikhonov et al., 1977) and the reconstruction problem is formulated as the constrained least squares minimization problem with the regularization parameter \( \lambda > 0 \) chosen properly (Zhang & Cherkaev, 2009; Zhang & Lamoureux et al., 2009):

\[
\min_c \{ \| S_R c - g_R \|^2 + \| S_I c - g_I \|^2 + \lambda \| c \|^2 \} \quad \text{s. t.} \quad -\infty < s_n < 0, \quad 0 < \frac{A}{s_n} < 1, \quad n = 1,2,\cdots,q.
\]

After recovery of the coefficient vector \( c \) of the rational function approximation \( \hat{G}(s) \), its decomposition into partial fractions, gives \([p,q]-Padé\) approximation of \( G(s) \). Then the Q-factor and phase velocity can be calculated using the above derived formulas.

**Results**

In the first example we consider inverse modeling of Q-factor for a standard linear solid (SLS) (Generalized Zener) model. The values of material strain-stress relaxation times were chosen from (Tal-Ezer et al., 1990) to calculate the synthetic complex modulus \( M(\omega) \) with five relaxation mechanisms to yield a constant \( Q = 100 \) at 50 data points (frequency range: 2–50Hz), \( \rho = 2000 \) kg/m\(^3\) and \( M_R = M_U - \delta M = 8 \) Gpa. The function \( G(s) \) in this analytic model has a five-term partial fractions form. \( A_n \) and \( s_n \) are reconstructed almost exactly in the case of \( q \geq 5 \) when there is no noise in the data. The recovered poles and residues are further used to covert the values of the strain-stress relaxation times with the number of relaxation mechanisms being less than five using the low order \([p,q]-Padé\) approximant method for evaluating Q-factor and phase velocity. Fig. 1 illustrates the numerical results for sensitivity analysis of the estimated Q-factors and phase velocities using data with added noise. The results agree with the published simulations in (Tal-Ezer et al., 1990). To further examine the effectiveness of the developed inversion method we consider a nearly constant \( Q = 21 \) model with a continuous relaxation spectrum. The complex velocity measurements were simulated at 50 data points (frequency range: 0.01–100Hz). Fig. 2 shows the estimation of Q-factors and phase velocities using Padé approximation for \( q = 4 \) and \( q = 5 \).

**Conclusions**

A new numerical inversion method for estimation of Q-factor and phase velocity in homogeneous dissipating media was developed using Padé approximation. The inverse problem was solved as a constrained least squares minimization problem with regularization.
constraints provided by the Stieltjes integral representation of the complex modulus. The method was tested using analytical models of viscoelastic media with a continuous spectrum as well as a standard linear solid (Zener) model. The numerical results demonstrate the effectiveness of the developed approach. The method can be used for identification of relaxation parameters of viscoelastic materials from measurements of complex velocity or complex modulus. The recovered relaxation mechanisms can be used for numerical modeling of seismic wavefields in viscoelastic media.

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References

Fig. 1 True and computed Q-factor (left) and phase velocity (right) for data with 1.0%, 1.5% and 2.5% noise for the standard linear solid (Zener) model (q = 5 in the inversion algorithm).

Fig. 2 Calculation of Q-factors (left) and phase velocity (right) for nearly constant Q =21 model with a continuous spectral measure using different orders of Padé approximation.